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VISUAL CRITERION FOR UNDERSTANDING THE NOTION OF CONVERGENCE IF INTEGRALS IN ONE PARAMETER

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Abstract: Admittedly, the notion of generalized integrals in one parameter has a fundamental role. En virtue that, in this paper, we discuss and characterize an approach for to promote the visualization of this scientific mathematical concept. We still indicate the possibilities of graphical interpretation of formal properties related to notion of convergence/divergence. Moreover, supported by technology, we show the relations between the class of functions $z = f(x, y)$ and $y = f(t, x)$ related to the two and three-dimensional spaces. At the end, we propose a visual criterion for the recognition and perception of graphical-dynamic properties provided by the Dynamic System Geogebra – DSG and the Computer Algebraic System – CAS Maple. Finally, our perspective indicates the visualization as an important cognitive component for the teaching and the learning at the academic level.

Key words: Visual criterion, Integrals in one parameter, Geogebra, CAS Maple.

1. Introduction

We start our discussion by an illustrative example. With this attention, we consider the function $f(x, y) = (2x + y^3)^2$. From the Multivariable Calculus, we know that its graph lies in the IR^3 (Tall, 1986). Traditionally, we can consider two kinds of restrictions. In fact, we can take the following expression $g(t, y) = (2t + y^3)^2$. By the same argument, we can consider too $h(x,t) = (2x + t^3)^2$. These two restriction allow us to analyze its graphical behavior in the IR^2. In fact, we show some properties in the figure 1.

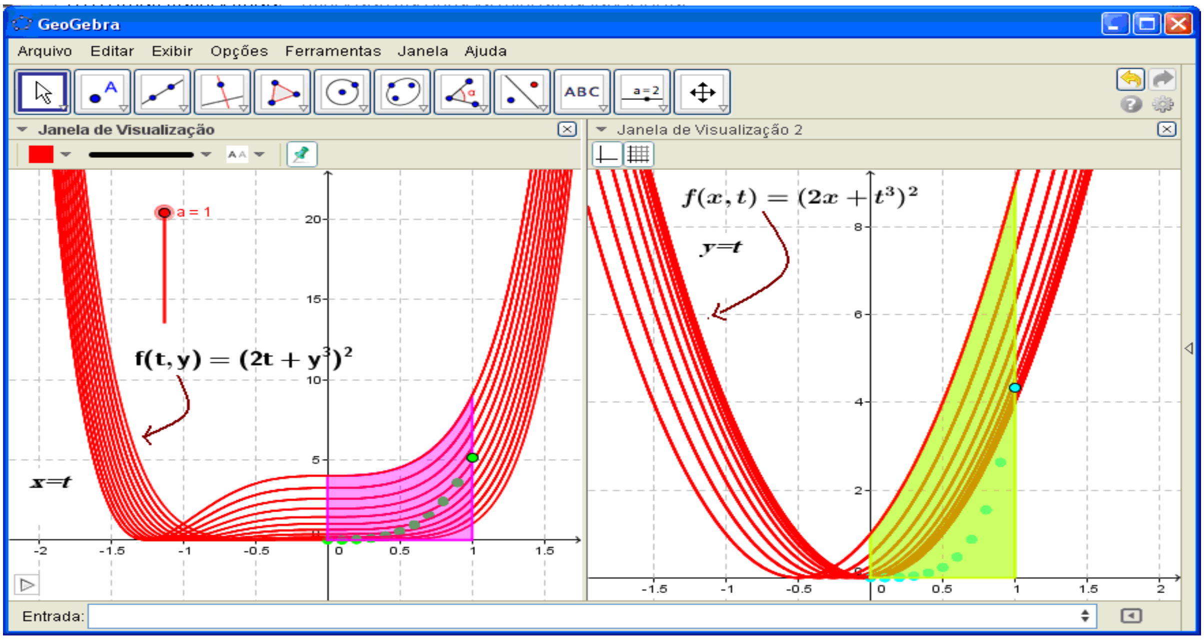

Figure 1. *Descrption of the graph´s behavior in the two dimensional space and it´s restrictions*

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When we restrict our action to the analytical treatment, we can still consider $\int_0^1 f(t,x)dx = \int_0^1 (2x+t^3)dx = \left[\frac{(2x+t^3)^3}{6}\right]_{x=0}^{x=1} = \frac{4}{3}+2t^3+t^6 = h(t)$. On the other hand, en virtue that we considered only integrable functions, we can visualize the graphical-geometric behavior corresponding the two restriction aforementioned. In the figure 1, we can understand that under the restriction $[0,1] \subset IR$ we observe a limited image for both restrictions. We still indicate the numerical behavior of the corresponding areas provided by Fundamental Theorem of Calculus – FTA.

Admittedly, the abstract character involved in this last analytical argumentation deserves a careful attention from the universities teachers (Tall, 1991). In fact, if we carefully analyze these last results, we note that $h(t)$ represents a function of t, which makes sense since the integrand depends on t. We must observe that we integrate over x and are left with something that depends only on t, and not depends of the variable x. We explore the technology with the aim of highlighting certain qualitative characters, such as: the convergence of generalized integrals; the behavior of the contributions of area. In the next section we discuss the notion of integrals that depend on a parameter t. We establish some formal properties related to the class of functions denoted by $f(t,x)$. Moreover, the Dynamic System Geogebra - DSG will provide to us the visualization.

2. Preliminary properties and some examples

Let us consider the class of the functions $F(t): t \mapsto \int_a^b f(t,x)dx$ or $F(t): t \mapsto \int_a^{+\infty} f(t,x)dx$, where $t \in I \subset IR$. We will indicate the two main theorems of this section. With these two theorems, we can conclude, from the formal point of view, something about the convergence behavior of each integral related to the function $F(t)$.

Theorem 1: Consider the function $F(t): t \mapsto \int_a^b f(t,x)dx$, where $\begin{array}{l} f: I\times[a,b) \to IR \\ (t,x) \mapsto \quad \text{f(t,x)} \end{array}$ and exist φ such that: $|f(t,x)| \le \varphi(x)$, for $\forall t \in I, \forall x \in [a,b)$. Moreover, we still have $\int_a^b \varphi(x)dx < \infty$. In these terms, we conclude that $F(t): t \mapsto \int_a^b f(t,x)dx$ is continuous.

Theorem 2: For $n \in IN$, consider the function $\frac{\partial^n f(t,x)}{\partial t^n}$ continue over $I \times [a,+\infty)$. If exists a function $\varphi(t)$ such that $\left|\frac{\partial f(t,x)}{\partial t}\right| \le \varphi(t)$. Moreover, if for each $x \mapsto \frac{\partial^n f(t,x)}{\partial t^n}$ is continuous over $[a,\infty) \subset IR$. Then, the function $F: t \in I \mapsto \int_a^\infty f(t,x)dx$ belongs to C^n and $F^{(i)}(t) = \int_a^\infty \frac{\partial^n f(t,x)}{\partial t^n}dx,\ 1 \le \text{i} \le \text{n}$.

When we assume a highly formalistic style, these two theorems are sufficient to obtain any results regarding all mathematical issues that we will discuss here. On the other hand, the technology promotes the production of tacit inferences arising from the perception of some complex mathematical objects. Moreover, regarding the integrals dependent on parameters, we can find some cases that involve a rapid analytical argument for its complete solution. In fact, let us consider the $F(t) = \int_0^1 sen(tx)dx$. In this case, we can establish that $F(t) = -[\cos(tx)/t]_{x=0}^{x=1} = 1-\cos(x)/t = h(t)$. However, we question: what is the graphical-geometric meaning related to this process? On the other hand, we will discuss some properties related

to these theorems. We will describe, preliminarily, some analytical methods for evaluate the integrals on parameters. Indeed, throughout the text we will seek to answer this and other questions.

Let us consider $A(t)=\left(\int_0^t e^{-x^2}dx\right)^2$. From this expression, we easily write $A'(t)=2\left(\int_0^t e^{-x^2}dx\right)\cdot e^{-t^2}=2e^{-t^2}\int_0^t e^{-x^2}dx$. Next, we use the substitution $x=ty$ and, from this, we write: $A'(t)=2e^{-t^2}\int_0^1 e^{-x^2}dx=2e^{-t^2}\int_0^1 te^{-t^2\cdot y^2}dy=\int_0^1 2te^{-(y^2+1)t^2}dy$. Then follows that $A'(t)=\int_0^1 2te^{-(y^2+1)t^2}dy=\int_0^1-\frac{\partial}{\partial t}[\frac{e^{-(y^2+1)t^2}}{1+y^2}]dy=-\frac{d}{dt}\int_0^1\frac{e^{-(y^2+1)t^2}}{1+y^2}dy$. From this last expression, we can assume that $B(t):=\int_0^1\frac{e^{-(x^2+1)t^2}}{1+x^2}dx$. Henceforth, we will get that $-B'(t):=-\frac{d}{dt}\int_0^1\frac{e^{-(x^2+1)t^2}}{1+x^2}dx=A'(t)\therefore A'(t)=-B'(t)\leftrightarrow A(t)=-B(t)+C$, for all $t>0$.

Now, we can get the value for $C\in IR$. For this, we take $t\to 0^+$ and, easily, we get that $A(t)=\left(\int_0^t e^{-x^2}dx\right)^2\overset{t\to 0^+}{\to}\left(\int_0^0 e^{-x^2}dx\right)^2=0$. On the other hand, we observe that $B(t):=\int_0^1\frac{e^{-(x^2+1)t^2}}{1+x^2}dx\overset{t\to 0^+}{\to}\int_0^1\frac{e^0}{1+x^2}dx=-\frac{\pi}{4}+C$. Finally, we obtain that $C=-\frac{\pi}{4}$. This numerical date is fundamental en virtue the following equality $A(t)=-B(t)+C\leftrightarrow$ $\left(\int_0^t e^{-x^2}dx\right)^2=\frac{\pi}{4}-\int_0^1\frac{e^{-(x^2+1)t^2}}{1+x^2}dx$. In the last stage of the verification, we assume $t\to+\infty$ and get that $J^2=\left(\int_0^\infty e^{-x^2}dx\right)^2=\frac{\pi}{4}-\int_0^1\frac{e^{-(x^2+1)t^2}}{1+x^2}dx\to\frac{\pi}{4}-0$. From this, we reach a numerical property $J^2=\left(\int_0^\infty e^{-x^2}dx\right)^2=\frac{\pi}{4}\therefore J=\frac{\sqrt{\pi}}{2}$.

Our second example is $I_n(t)=\int_0^{+\infty}\frac{dx}{(x^2+t^2)^n}$. In this case, we take $a,A\in IR$ and the function $F_n(t,x)=\frac{1}{(t^2+x^2)^n}$ and observe the $F_n(t,x)$ is continuous in the interval $[0,+\infty)$. Moreover, we infer that $F_n(t,x)\overset{t\to+\infty}{\sim}\frac{1}{x^{2n}}>0$ (we show this equivalence relation in the figure 2 through the comparison of areas). As regards its partial derivative, we obtain $\frac{\partial F}{\partial t}(t,x)=-\frac{2nt}{(t^2+x^2)^{n+1}}$, for all $(t,x)\in[a,A]\times[0,+\infty)$. From this equality, we establish $\left|\frac{\partial F}{\partial t}(t,x)\right|=\left|\frac{2nt}{(t^2+x^2)^{n+1}}\right|\le\frac{2\cdot n\cdot A}{(t^2+a^2)^{n+1}}=\varphi(t)$. From the last data, we infer:

$I_n(t)=\int_0^{+\infty}\frac{dx}{(x^2+t^2)^n}\therefore I'_n(t)=\frac{\partial}{\partial t}\int_0^{+\infty}\frac{dx}{(x^2+t^2)^n}=\int_0^{+\infty}\frac{d}{dt}\frac{dx}{(x^2+t^2)^n}=-2nt\int_0^{+\infty}\frac{dx}{(x^2+t^2)^{n+1}}=-2nt\cdot I_{n+1}(t)$. From this, we establish the relation $I'_n(t)=-2nt\cdot I_{n+1}(t)$ (*). We can take some particular cases, like $I_1(t)=\int_0^{+\infty}\frac{dx}{(x^2+t^2)^1}=\frac{1}{t}\left[\arctan\left(\frac{x}{t}\right)\right]_{x=0}^{x=+\infty}=\frac{\pi}{2t}=\frac{I'_0(t)}{-2t}\therefore I'_0(t)=-\pi$ and $I_1(t)=\frac{\pi}{2t}$.

From last equality, we obtain that $-\frac{\pi}{2t^2}=I'_1(t)=-2t\cdot I_2(t)\therefore I_2(t)=\frac{\pi}{4t^3}$. We still compute $I'_2(t)=-\frac{\pi}{12t^4}\therefore-\frac{\pi}{12t^4}=I'_2(t)=-4t\cdot I_3(t)\leftrightarrow I_3(t)=\frac{\pi}{48t^5}$. Once again, we obtain that $I'_3(t)=-\frac{\pi}{240t^6}\therefore I'_3(t)=-6t\cdot I_4(t)$. So, we write $I_4(t)=\frac{\pi}{1440t^7}$. En virtue the equality (*), we compute $I'_4(t)=-\frac{\pi}{10080t^8}\therefore I_5(t)=\frac{\pi}{80640t^8}$. We can generalize these properties and, ultimately, to establish that $I_n(t)\overset{n\to\infty}{\longrightarrow}0$ ($t\in IR$). We comprehend this limit in the figure below!

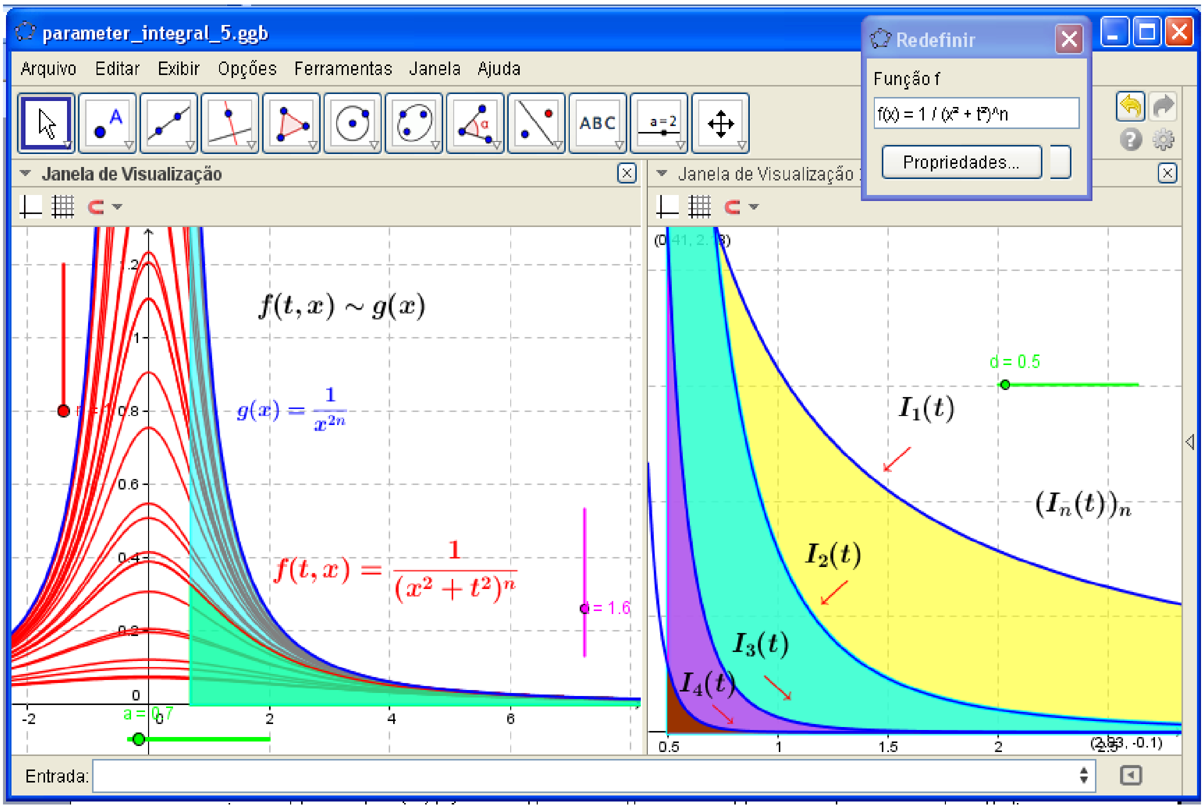

Figure 2. *Graphical description to the generalized integral in one parameter and the area's contributions*

We take other example $I(t)=\int_0^{\infty}\frac{\cos(xt)}{x+t}dx$, with condition that $t>0$. The obvious inequality is $\left|\frac{\cos(xt)}{x+t}\right|\le\frac{1}{x+t}$. However, this inequality is useless, in virtue that $\int_0^{\infty}\frac{1}{x+t}dx$ diverges! On the other hand, integration by parts yields

$\int_r^{r_1} \frac{\cos(xt)}{x+t}dx = \frac{sen(xt)}{t(x+t)}\Big|_{x=r}^{x=r_1} + \int_r^{r_1} \frac{\cos(xt)}{t(x+t)^2}dx = \frac{sen(rt)}{t(r+t)} - \frac{sen(r_1 t)}{t(r_1+t)} + \int_r^{r_1} \frac{\cos(xt)}{t(x+t)^2}dx$. Therefore, if we take $0 < r < r_1$, then: $\left|\int_0^{\infty} \frac{\cos(xt)}{x+t}dx\right| \le \left|\frac{sen(rt)}{t(r+t)} - \frac{sen(r_1 t)}{t(r_1+t)} + \int_r^{r_1} \frac{\cos(xt)}{t(x+t)^2}dx\right| \le$

$< \frac{1}{t(r+t)} + \frac{1}{t(r_1+t)} + \int_r^{r_1} \frac{1}{t(x+t)^2}dx = \frac{2}{t(r+t)} + \int_r^{r_1} \frac{1}{t(x+t)^2}dx \le \frac{3}{t(x+t)^2} = h(t,x)$. From this last inequality, we can compare the numerical behavior between $\left|\int_0^{\infty} \frac{\cos(xt)}{x+t}dx\right|$ and the function $h(t,x) = \frac{3}{t(x+t)^2}$. In all these examples, we use an argument of logical-formal character. In fact, is evident from these elements, we have reduced several chances to explore the perception and visualization of certain properties related to the notion of convergence.

Finally, we consider the function $f(t,x) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if (t,x)} \neq (0,0) \\ 0 \text{ if (t,x)=(0,0)} \end{cases}$. Easily, we indicate a other function $f(x,y) = \begin{cases} \frac{yx^3}{(y^2+x^2)^2} & \text{if (x,y)} \neq (0,0) \\ 0 \text{ if (x,y)=(0,0)} \end{cases}$. We evaluate the integral $F(t) = \int_0^1 \frac{xt^3}{(x^2+t^2)^2}dx$ and still observer that $F(0) = \int_0^1 \frac{0}{(x^2+t^2)^2}dx = 0$. However, for $t \neq 0$ we determine $F(t) = \int_0^1 \frac{xt^3}{(x^2+t^2)^2}dx \underset{du=2xdx}{\overset{u=x^2+t^2}{=}} \int_{t^2}^{1+t^2} \frac{t^3}{2u^2}du = -\left[\frac{t^3}{2u}\right]_{u=t^2}^{u=1+t^2} = \frac{t}{2(1+t^2)}$. Henceforth, we can get $F'(t) = \frac{1-t^2}{2(1+t^2)^2}$. Now, we can compute the following derivative $\frac{\partial f}{\partial t}(t,x)|_{x\neq 0} = \frac{(x^2+t^2)^2(3xt^2) - xt^3(x^2+t^2)2t}{(x^2+t^2)^4} = \frac{xt^2(3x^2-t^2)}{(x^2+t^2)^3}$.

En virtue to describe its behavior, we will write $\frac{\partial f}{\partial t}(t,x) = \begin{cases} \frac{xt^2(3x^2-t^2)}{(x^2+t^2)^3} & \text{if x} \neq 0 \\ 0 \text{ if x=0} \end{cases}$. As we indicated earlier, this function if the constraint of another function $\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{yx^2(3y^2-x^2)}{(x^2+y^2)^3} & \text{if x} \neq 0 \text{ or y} \neq 0 \\ 0 \text{ if x=0=y} \end{cases}$. We indicate that the analytical character of these arguments, not aid an intuitive understanding of the mathematical process. We note that $F'(0) = \frac{1}{2(1+0^2)^2} = \frac{1}{2}$. On the other hand, we write $\frac{d}{dt}\int_0^1 f(t,x)dx = \int_0^1 \frac{\partial}{\partial t} f(t,x)dx = \int_0^1 0dx = 0 \neq \frac{1}{2}$. This fact represents a counterexample related to the differentiating under the integral sign (Leibniz´s rule). (see fig. 6).

3. Some examples with the Dynamic System Geogebra – DSG

In this section, we will emphasize the graphic-geometric interpretation provided by the DSG. Thus, from certain arguments and inequalities shown in the previous section, we will indicate possibilities for its heuristic interpretation (Alves, 2014a, 2014b), in each occasion we use the software. In fact, in the figure 2, we indicate the regions under the graph of the functions $f(x) = e^{-x^2/2}$, $g(x) = e^{-x^2}$ and $h(x) = e^{-\pi x^2}$. We acquire a tacit feeling about the relationship of the contributions of area under these curves.

In this figure, we consider the following integrals $I = \int_{-\infty}^{+\infty} e^{-x^2/2} dx$, $J = \int_{0}^{+\infty} e^{-x^2} dx$ and $K = \int_{-\infty}^{+\infty} e^{-\pi x^2} dx$. For the $x > 0$ we see that $K \le J \le I$. Relatively to these integrals, we can prove the formal properties $J = \frac{I}{2\sqrt{2}}$ e $K = \frac{I}{\sqrt{2\pi}}$. We have showed in the previous se sections $J = \frac{\sqrt{\pi}}{2}$. From this, we compute the numerical values $I = J \cdot 2\sqrt{2} = \sqrt{2\pi} \therefore K = 1$. We still see the region correspondently the integral $B(t) := \int_0^1 \frac{e^{-(x^2+1)t^2}}{1+x^2} dx$. So, we can conjecture about the contribution's area restricted in $t \in [0,1]$. For some values $0 < t$ we obtain the behavior of a family of functions $B(t)$. In the left side, we can explore a global and local graphical-geometric behavior to these integrals.

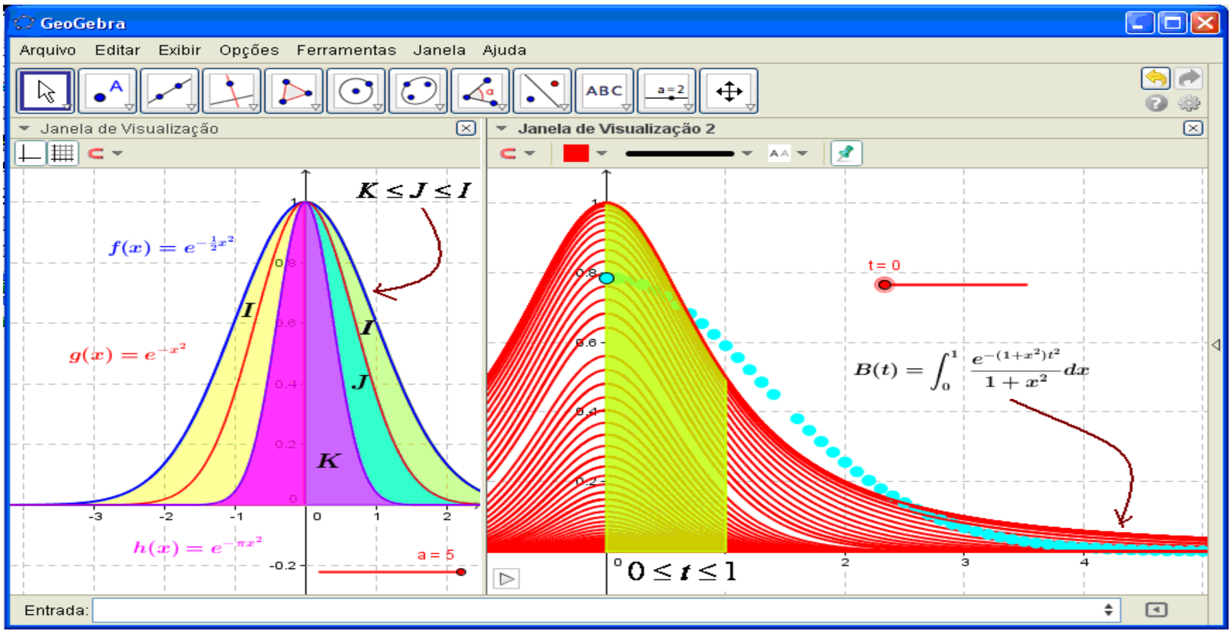

Figure 3. *The visualization of multiple elements related to the convergence in one parameter*

We have analyzed the integral $I(t) = \int_0^{\infty} \frac{\cos(xt)}{x+t} dx$. In the figure 3, we can develop a local and global analysis related to the behavior for the family $f(t,x) = \frac{\cos(xt)}{x+t}$. In fact, on the left side (for $x < 0$) we note a divergence region. Visually, we could conjecture that the integral $\int_{-\infty}^{0} \frac{\cos(xt)}{x+t} dx$ diverges.

However, on the right side, the area´s contributions are decreasing. We highlight this behavior from the view of a green movel point (on the left side). En virtue the final conclusion about its behavior, we used the function $h(t,x)=\dfrac{3}{t(x+t)^2}$. Now, we can understand the inequality $\left|\int_0^{\infty}\dfrac{\cos(xt)}{x+t}dx\right|\leq\dfrac{3}{t(x+t)^2}=h(t,x)$ from the graphical-geometric point of view. From the figure 3, that the contribution's area are limited. This behavior is in line with the elements of order logic that indicated in the previous section. Moreover, we encourage a perceptual ability in order to analysis the integrals.

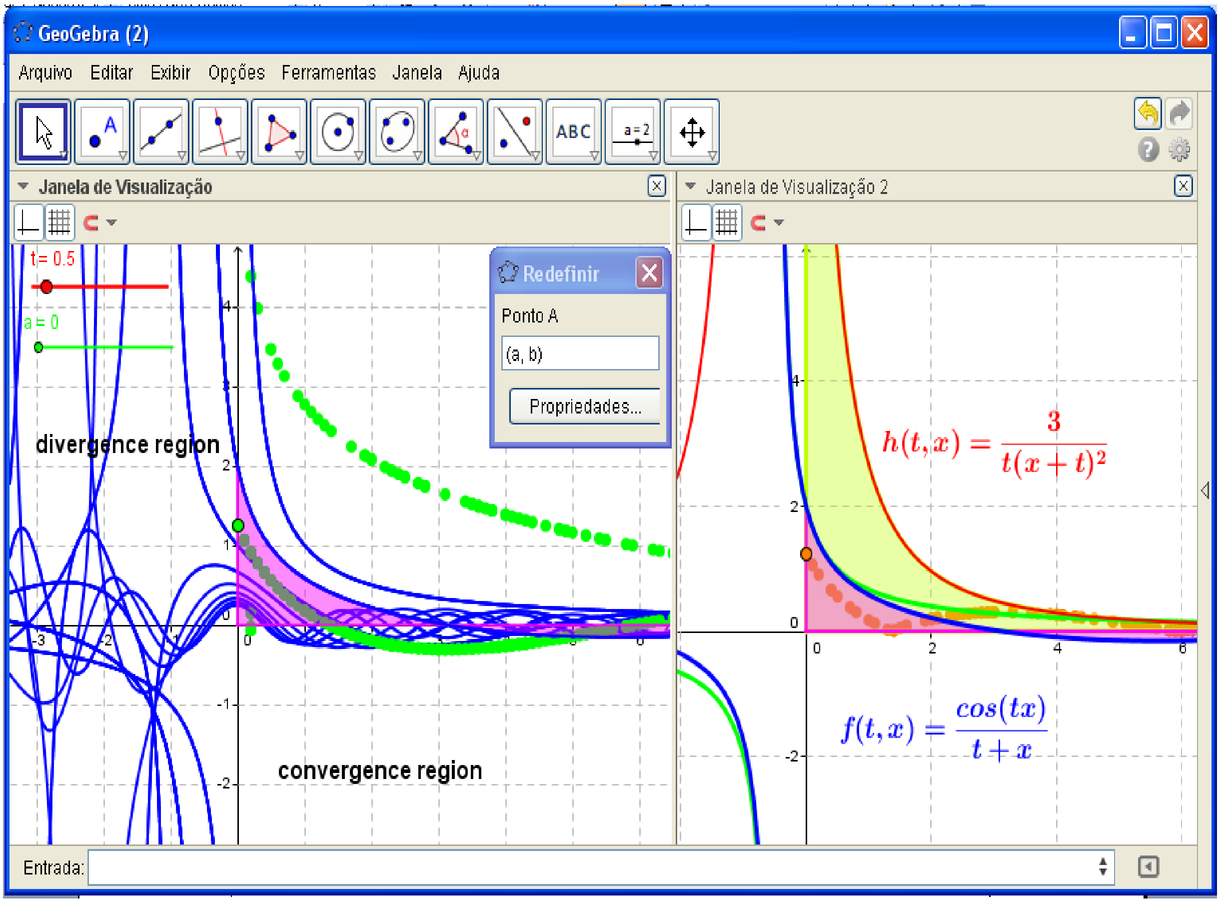

Figure 4. *The local and global analysis of the integrals in one parameter*

In the next section, we will highlight the links between convergence in the plane and convergence in space IR^3. The first is related to the area's contributions. While the second we will relate to the volume's contributions. We will show that the software Maple provides essential relations in the context of teaching. From these elements, we will establish a visual criterion for analyze the convergence.

4. A visual criterion for analyse the convergence of integrals

Now, we use the CAS Maple, with the intention to acquire an understanding related to the mathematical process which we have discussed in the previous sections. In fact, we visualize the surface described by the function $f(x,y)=\dfrac{yx^3}{(y^2+x^2)^2}$ restricted in the rectangular region $[0,3]\times[0,3]\subset IR^2$. On the other hand, we observe the graphics, related to the set indicated

by $(x, y, \frac{yx^3}{(y^2+x^2)^2}) \in IR \times IR \times IR$. We also determined, from the point of view of each constraint of the parameters, the corresponding parameterized curves (in red and blue colors) to such variations.

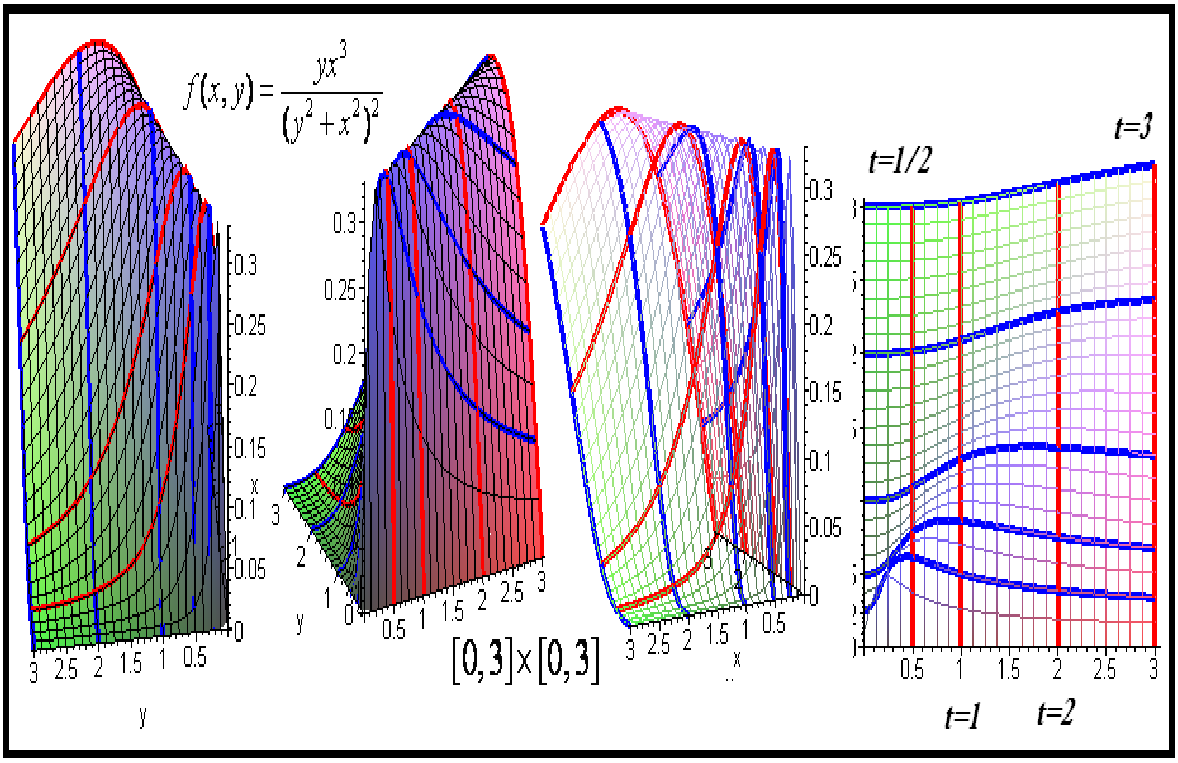

Figure 5. *The visualization of the constraint of the parameters* $t \in IR$ *in red and blue colors provided by CAS Maple*

In the figure 6, we can understand, from a graphic-geometric standpoint, the role of the counter-example we showed in the previous section. Indeed, we observed that at the origin the function has a problem with the expression $(x^2+y^2)^3=0$. The same effect we wait for the derivative $\frac{\partial f}{\partial t}(t,x)$. This counter-example shows when the differentiating under sign the integral fails (see discontinuity at the origin).

But, when we consider polynomial functions like $\int_0^1 f(t,x)dx = \int_0^1 (2x+t^3)^2 dx$ we always count with this rule. Indeed, we compute $\frac{d}{dt}\int_0^1 f(t,x)dx = \int_0^1 \frac{\partial}{\partial t}(2x+t^3)^2 dx = \int_0^1 2(2x+t^3)(3t^2)dx = 6t^2+6t^5$. This analytical rule can be re-signified from the complementary use of these softwares. In this case, for example, we can visualize the graphical behavior of the function $h(t)=6t^2+6t^5$ which, for all fixed $0 \le t \le 1$ represents the numerical value of an area.

With this attention, as a teacher, we can add other meanings for the complex derivation rule known by Leibniz rule (differentiation of integrals). Its verification can be find in Strichartz (2000, p. 432-433), for example. The key ideas employed in it are similar as in other books. However, when we consider some cognitive components in the learning context, we have to change our mathematical approach.

For example, we have commented some properties related to the function $\frac{\partial f}{\partial x}(x,y)=\frac{yx^2(3y^2-x^2)}{(x^2+y^2)^3}$.

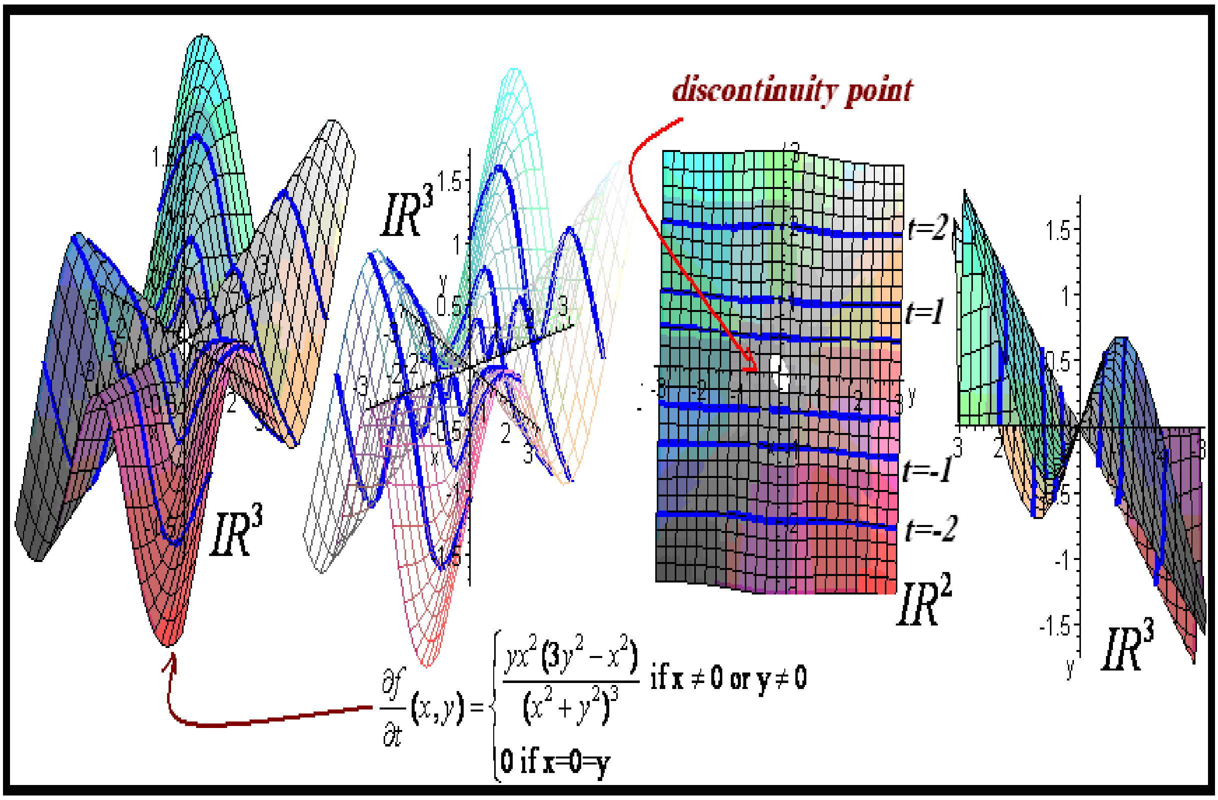

Figure 6. *Visual identification of the discontinuity at the origin* $(0,0,0)$ *provided by CAS Maple*

For example, if we take the function $f(x,y)=e^{-(x^2+y^2)}$. In the figure 7, we indicate in the IR^3 a surface related to this expression. We perceive that contribution's areas are progressively decreasing, for the values $x\rightarrow\infty$ and $y\rightarrow\infty$ (7-III). Indeed, we can attribute a geometrical meaning for the integral $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$. However, by the using of the polar coordinates, we get that $\sqrt{\pi}/2\cdot\sqrt{\pi}/2=J^2=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$.

We can compute the volume of the region below the surface and above the $xy-plane$ in several ways. However, we are interesting in the relation the contributions of volume and areas. En virtue that, in the figures 7-II and 7-III, we visualize the behavior in the $xz-plane$. These elements can contribute for us to conjecture the convergence of the generalized integrals. In the figure 5, 6 and 7, we indicate the blue curves which, in the IR^2 constituted the parameter t.

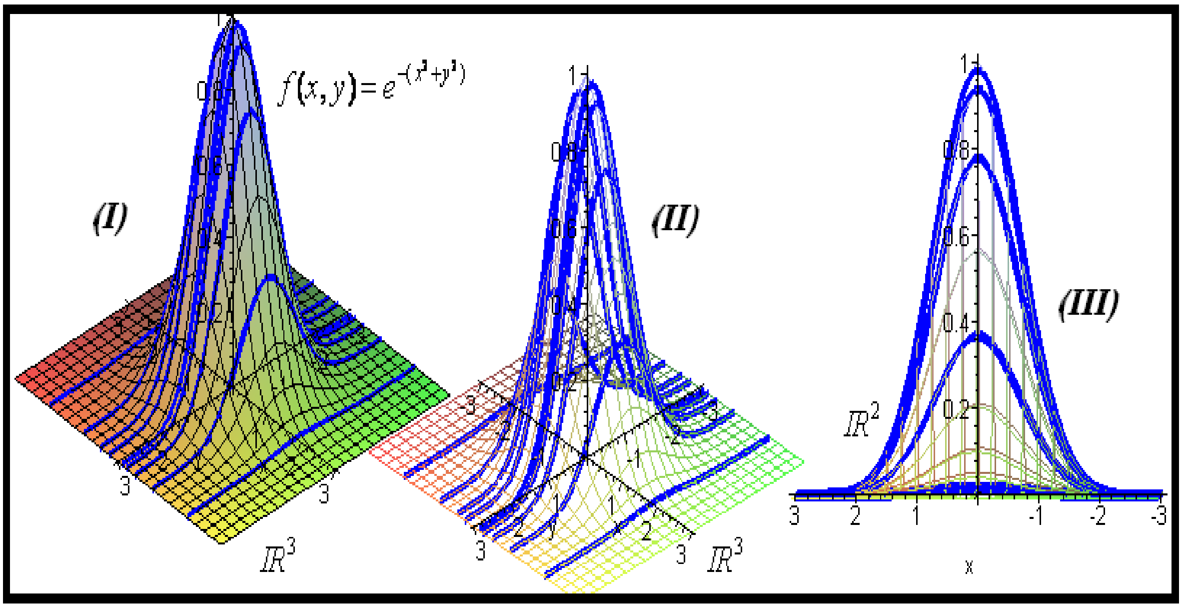

Figure 7. *The behavior related to the integral in the* IR^2 *and* IR^3 *provided by CAS Maple*

5. Final remarks

Undoubtedly, we observe the relevant place for generalized (or improper) integral at the academic locus. In the specific way, we discussed in this paper, several properties related to the integrals on parameters indicated by $\int_a^{\infty} f(t,x)dx$. We showed some structured examples supported by the visualization and perception of the dynamic-graphical properties. On the other hand, the formal and mathematical characters become relevant for to comprehend the convergence/divergence behavior of this integral.

Our use of the two mathematical softwares permits identify some limitations and qualitative properties related to the link between IR^2 and IR^3. So, we emphasize the follow points: (i) the didactical use of the DSG and the CAS Maple allows to relate the area's contributions with volume's contributions (see figure 7); (ii) the didactical use of the DSG and the CAS Maple permits a re-signification of the analytical and formal inequalities from the visual (and tacit) point of view; (iii) the didactical use of the DSG and the CAS Maple permits the production of conjectures about the behavior of the integrals in one parameter from the two dimensional space $\left(IR^2\right)$ (see figures 3 and 4); (iv) the didactical use of the DSG and the CAS Maple permits to the teacher indicate for the students the conceptual link between the class of function $f(t,x)$ and $f(x,y)$ (see figures 5 and 6); (v) the convergence/divergence related to some generalized integrals can be inferred directly to the exploration of the dynamic graph provided by DSG without a extreme formalism (see figure 2, 3 and 4).

Finally, we question a teaching that emphasizes only algorithmic and formal techniques that aim to verify the behavior of the integrals studied in previous sections (Alves, 2014c). In this formalistic standard approach, the theorems stated in this paper are the only way to reach conclusive results. On the other hand, when we are interested in learning, we can not just appreciate only logical results and definitive answers, but also the responses and arguments that lead to a new and unexpected mathematical investigation (Alves, 2012; Alves; Borges Neto & Ingar, 2013).

We described, with the aid of DGS, elements that enable the understanding on the integration process involving the functions depending on parameter. As well as, the process of moving from the differentiating under the integral sign. Formally, we have a function $\varphi(t)=\int_a^b f(t,x)dx$ and

compute $\frac{\partial \varphi}{\partial t}(t,x) = \int_a^b \frac{\partial f}{\partial t}(t,x)dx$ (Shakarchi, 1998, p. 225). We have used this rule in the some examples showed in this paper.

Finally, in the academic level, especially in their early years of study, we suggest a mathematical approach based in a moderate formalism (Tall, 1986; 1991). In this way, the heuristics and tacit components can be valued by the exploitation of the visualization. So, these introductory elements should guide a journey that involves later a greater formalization. This last fact characterizes the ritualistic style academic. In this context, we recognize the effort of some researchers with intention to show the advantages of using technology (Miller, 2013; Murillo, 2004; Nguyen, 2012).

Similarly our didactical and methodological effort, Zlatanov (2013) shows a complementary use of the two softwares which we have discussed here. On the other hand, this article shows some basic functions of DSG GeoGebra so that the readers may familiarize themselves with the program. The dynamical aspects of this software can motivate the students at academic level (Alves, 2012; 2013). And regarding to the mathematical academic teacher, with the use of actually technology, it can devise different approaches to some mathematical topics as we discussed here.

References

[1] Alves, F. R. V. (2014a). Visualizing with dynamic system Geogebra: the Fundamental Theorem of Algebra - TFA and its applications. *GGIJRO - Geogebra International Journal of Romania*, p. 39-50. Available in: http://ggijro1.files.wordpress.com/2014/01/art48.pdf

[2] Alves, F. R. V. (2014b). Visualizing the behavior of infinite series and complex power series with the Geogebra. *GGIJRO - Geogebra International Journal of Romania*, p. 41-51. In press.

[3] Alves, F. R. V. (2014c). La visualization de regiones em coordenadas polares y la determinación de aréa con el software Geogebra. *Revista Premisa*. Buenos Aires. Argentina. v. 59. nº 16. Available in: http://www.soarem.org.ar/publicaciones.html

[4] Alves, F. R. Vieira. (2013). Viewing the roots of polynomial functions in complex variable: the use of Geogebra and the CAS Maple. *Acta Didactica Naposcencia*. Romania, v, 6, nº 3, 58-45.

[5] Alves, F. R. V. (2012). Internal transition of Calculus: a discussion based on the Geogebra´s use in the Multivariable Calculus. *IJGGSP – International Journal of Geogebra of São Paulo*. v. 2, nº 1. 5-19. Available in: http://revistas.pucsp.br/index.php/IGISP/issue/view/895

[6] Alves, F. R. V; Borges Neto, H; Vigo Ingar, Kátia. (2013). Transición interna del Cálculo: una propuesta para la identrificación de elementoas de ruptura y de transición apoyada en la Sequencia Fedathi. *Revista Premisa*. Available in: http://www.soarem.org.ar/articulos.html

[7] Miller, David. (2013). Investigating the Generalization of a Special Property of Cubic Polynomials to Higher Degree Polynomials. *North American Geogebra Journal. Available* v. 2, nº 1. 5-9. in: http://www.ggbmidwest.com/ojs-2.3.4/index.php/ggbj/article/view/25

[8] Murillo, Rosa. Elvira, P. (2004). Procesos de construcción del concepto de límite en un ambiente de aprendizaje cooperrativo, debate científico y autorreflexión (tesis). Mexico: Universidad del Mexico. Available in: https://uacm.academia.edu/ROSAPAEZ/TESIS-DE-DOCTORADO.

[9] Nguyen, Dahn. Nam. (2012). Geogebra with an interactive help system generates abductive argumentation during proving process. *North American Geogebra Journal. Available* v. 1, nº 1. 1-5. in: http://www.ggbmidwest.com/ojs-2.3.4/index.php/ggbj/article/view/21

[10] Shakarchi, Rami. (1998). Problems and solutions for undergraduate Analysis. Princeton: Princeton University.

[11] Strichartz, Robert. (2000). The way of Analysis. London: Jones and Barttle Pubishers. Revised edition.

[12] Tall, David. (1986). Drawing implicit functions. *Mathematics in the School*. nº 15, 2, 33-37.

[13] Tall, David. (1991). Intuition and Rigor: the role of visualization in the Calculus. *Visualization in Mathematics*. nº 19, 105-119.

[14] Zlatanov, Boyan. (2013). Some properties of Reflection of Quadrangle about Point. *Annals. Computer Sciences Series*. v. 11. nº 1. 79-91. Available in: http://anale-informatica.tibiscus.ro/download/lucrari/11-1-11-Zlatanov.pdf

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